

A constructive test for exponential stability of linear time-varying discrete-time systems

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Abstract We complete the stability results of the paper Bourlès et al. (SIAM J Control Optim 53:2725–2761, 2015), and for this purpose use the linear time-varying (LTV) discrete-time behaviors and the exponential stability (e.s.) of this paper. In the main theorem we characterize the e.s. of an autonomous LTV system by standard spectral properties of a complex matrix connected with the system. We extend the theory of discrete-time LTV behaviors, developed in the quoted publication, from the coefficient field of rational functions to that of locally convergent Laurent series or even of Puiseux series. The stability test can and has to be applied in connection with the construction of stabilizing compensators.

Keywords Exponential stability · Discrete-time behavior · Linear time-varying

Mathematics Subject Classification 93D20 · 93C55 · 93C05

1 Introduction

We complete the stability results of [4] and use the notions, in particular the linear time-varying (LTV) discrete-time behaviors and the exponential stability (e.s.), of this paper, but extend the theory from the coefficient field $\mathbb{C}(z)$ of *rational functions* to the larger field $\mathbb{C} \ll z \gg$ of *locally convergent* Laurent series with at most a pole at 0. In the main Theorem 1.1 together with Corollary 3.11 we characterize e.s. of an autonomous LTV system by standard spectral properties of a complex matrix connected with the system. Due to, for instance, [6, 14], Theorem 1.1 furnishes a *constructive* test for e.s.

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This test can and has to be applied in connection with the construction of stabilizing compensators; cf. [12] in the case of differential LTV systems.

The proof of Theorem 1.1 on state space behaviors is contained in Sect. 2. Section 3 presents the theory of discrete-time LTV behaviors for the coefficient field $\mathbb{C} \langle\langle z \rangle\rangle$, but we expose the necessary modifications of [4] only. In particular, behaviors and their morphisms, autonomous behaviors and their e.s. are recalled from [4] and adapted. Essential properties of these are stated in Corollary 3.10 on module-behavior duality, in Corollary 3.12 on closure properties of the class of e.s. autonomous behaviors and in Corollary 3.11. The latter enables the application of Theorem 1.1 to arbitrary autonomous behaviors instead of state space behaviors only.

In Sect. 4 we shortly extend the results to the still larger field of *locally convergent Puiseux series* (cf. [16], [5, §3.1]). The latter field seems to be the largest coefficient field for which a reasonable stability theory for *general LTV systems* can be developed. We refer to the books [15, pp. 423–461] and [8, pp. 193–368] for comprehensive surveys of exponential stability of *state space systems*. Part II of the book [3] contains a detailed theory of general LTV behaviors and their stability that was modified in the papers [4, 5]. We also refer to the recent papers [1, 2, 7, 10, 13].

Theorem 1.1 requires some preliminary explanations: let $\mathbb{C} \langle\langle z \rangle\rangle$ denote the local principal ideal domain of *locally convergent power series* in the variable z and $\mathbf{K} := \mathbb{C} \langle\langle z \rangle\rangle$ its quotient field. A formal power series $a = \sum_{i=0}^{\infty} a_i z^i$ is called *locally convergent* if

$$\sigma(a) := \limsup_{i \in \mathbb{N}} \sqrt[i]{|a_i|} < \infty. \text{ Then } \rho(a) := \sigma(a)^{-1} \quad (1)$$

is its *convergence radius* and $a(z)$ is holomorphic in the open disc

$$\mathbf{D}(\rho) := \{z \in \mathbb{C}; |z| < \rho\}, \quad \rho := \rho(a). \quad (2)$$

The power series a is a unit (invertible) if and only if $a(0) = a_0 \neq 0$, and z is the unique prime of $\mathbb{C} \langle\langle z \rangle\rangle$, up to units. Each nonzero $a \in \mathbb{C} \langle\langle z \rangle\rangle$ has a unique representation

$$a = z^k b, \quad k := \text{ord}(a) := \min \{i; a_i \neq 0\} \in \mathbb{N}, \quad b = \sum_{j=0}^{\infty} a_{j+k} z^j \text{ invertible.} \quad (3)$$

This implies that every nonzero $a \in \mathbf{K}$ has the analogous unique representation

$$a(z) = z^k b = \sum_{j=k}^{\infty} b_{j-k} z^j, \quad \text{ord}(a) := k \in \mathbb{Z}, \quad b := \sum_{i=0}^{\infty} b_i z^i, \quad b_0 = b(0) \neq 0 \quad (4)$$

$$\sigma(a) := \sigma(b) := \limsup_{i \in \mathbb{N}} \sqrt[i]{|b_i|} < \infty, \quad \rho(a) := \sigma(a)^{-1}.$$

The element a is a power series if and only if $k = \text{ord}(a) \geq 0$. The representation $a = \sum_{j=k}^{\infty} b_{j-k} z^j$ shows that a is a locally convergent Laurent series with at most

a pole at zero, and that indeed $\mathbf{K} = \mathbb{C} \langle\langle z \rangle\rangle$ consists of all these Laurent series. If $k < 0$ the function $a(z)$ is a holomorphic function in the *pointed* open disc $\mathbf{D}(\rho(a)) \setminus \{0\}$. In particular, the function $a(t^{-1}) = t^{-k}b(t^{-1})$ is a smooth function on the real open interval $(\sigma(a), \infty) := \{t \in \mathbb{R}; t > \sigma(a)\}$, hence

$$a(t^{-1}) \in C^\infty(\sigma(a), \infty) := \{f : (\sigma(a), \infty) \rightarrow \mathbb{C}; f \text{ smooth}\}. \quad (5)$$

The sequences $a(t^{-1})$, $t \in \mathbb{N}$, $t > \sigma(a)$, are the time-varying coefficients of the difference equations of the present paper that in [4] were used for $a \in \mathbb{C}(z) = \mathbb{C}(z^{-1})$. The coefficient functions $a(t^{-1})$, $a \in \mathbf{K}$, are of at most polynomial growth on each closed interval $[\sigma_1, \infty)$, $\sigma_1 > \sigma(a)$, i.e., there are $c > 0$ and $m \in \mathbb{N}$ such that $|a(t^{-1})| \leq ct^m$ for $t \geq \sigma_1$ [5, (29)]. For nonzero a there is $\sigma_2 > \sigma(a)$ such that $a(t^{-1}) \neq 0$ for $t \geq \sigma_2$. These properties of the coefficient functions are essential for the module-behavior duality and for the definition and properties of exponential stability of autonomous behaviors.

Let, more generally, $A = (A_{\mu\nu})_{1 \leq \mu, \nu \leq n} \in \mathbf{K}^{n \times n}$ be any square matrix and define

$$\sigma(A) := \max \{ \sigma(A_{\mu\nu}); 1 \leq \mu, \nu \leq n \}, \quad \rho(A) := \sigma(A)^{-1}. \quad (6)$$

Then the function $t \mapsto A(t^{-1})$ is a smooth matrix function on the open real interval $(\sigma(A), \infty)$. For $t_0 \in \mathbb{N}$ we consider the signal space

$$W(t_0) := \mathbb{C}^{t_0 + \mathbb{N}} := \{(w(t))_{t \geq t_0}; t \in \mathbb{N}, w(t) \in \mathbb{C}\} \quad (7)$$

of complex sequences or discrete signals starting at the initial time t_0 . For $n \in \mathbb{N}$ we use the column spaces \mathbb{C}^n and $W(t_0)^n$ and identify

$$W(t_0)^n = (\mathbb{C}^n)^{t_0 + \mathbb{N}} \ni (w_1, \dots, w_n)^\top = (w(t_0), w(t_0 + 1), \dots), \quad w_i(t) = w(t)_i. \quad (8)$$

If $t_0 > \sigma(A)$, $t_0 \in \mathbb{N}$, the matrix A gives rise to the *state space equation* resp. the *behavior* or *solution space*

$$\begin{aligned} x(t+1) &= A(t^{-1})x(t), \quad t \in \mathbb{N}, \quad t \geq t_0, \quad \text{resp.} \\ \mathcal{K}(A, t_0) &:= \left\{ x \in W(t_0)^n; \forall t \geq t_0 : x(t+1) = A(t^{-1})x(t) \right\}. \end{aligned} \quad (9)$$

The *transition matrix* [15, p. 392] associated to (9) is

$$\Phi_A(t, t_0) := A((t-1)^{-1}) * \dots * A(t_0^{-1}), \quad t \geq t_0 > \sigma(A), \quad \Phi_A(t_0, t_0) = \text{id}_n. \quad (10)$$

There is the obvious isomorphism

$$\mathcal{K}(A, t_0) \cong \mathbb{C}^n, \quad x \mapsto x(t_0), \quad x(t) = \Phi_A(t, t_0)x(t_0). \quad (11)$$

For $\xi = (\xi_1, \dots, \xi_n)^\top \in \mathbb{C}^n$ and $M \in \mathbb{C}^{n \times n}$ we use the maximum norms

$$\|\xi\| := \max_i |\xi_i| \text{ and } \|M\| := \max \{ \|M\xi\|; \xi \in \mathbb{C}^n, \|\xi\| = 1 \}. \quad (12)$$

The system, i.e., the matrix A and the equation and behavior from (9), are called *exponentially stable* (e.s.) [4, Def. 1.7, Cor. 3.3] if

$$\begin{aligned} \exists t_0 > \sigma(A) \exists \alpha > 0 \exists \text{ p.g. } \varphi \in \mathbb{C}^{t_0 + \mathbb{N}}, \varphi > 0, \\ \forall t \geq t_1 \geq t_0 : \|\Phi_A(t, t_1)\| \leq \varphi(t_1) e^{-\alpha(t-t_1)}. \end{aligned} \quad (13)$$

Here a sequence $\varphi \in \mathbb{C}^{t_0 + \mathbb{N}}$ is called of at most polynomial growth (p.g.) if $|\varphi(t)| \leq ct^m$, $t \geq t_0$, for some $c > 0$ and $m \in \mathbb{N}$. In [4] we used the notation $\rho := e^{-\alpha} < 1$ and $\rho^{t-t_1} = e^{-\alpha(t-t_1)}$. In particular, e.s. implies asymptotic stability, i.e., $\lim_{t \rightarrow \infty} \Phi_A(t, t_1) = 0$ for $t_1 \geq t_0$. The system is called *uniformly e.s.* (u.e.s.) [15, Def. 22.5] if φ in (13) can be chosen constant. Notice that (13) is a property of the behavior family $(\mathcal{K}(A, t_1))_{t_1 \geq t_0}$ and of the trajectories $x(t) = \Phi_A(t, t_1)x(t_1)$, $t \geq t_1 \geq t_0$, for sufficiently large t_0 . This is appropriate for stability questions where the behavior of $x(t)$ for $t \rightarrow \infty$ is investigated. In [15] the author considers LTV state space equations $x(t+1) = F(t)x(t)$, $t \geq 0$, with an arbitrary sequence of complex matrices $F = (F(0), F(1), \dots) \in (\mathbb{C}^{n \times n})^{\mathbb{N}} = (\mathbb{C}^{\mathbb{N}})^{n \times n}$. All stability results in [15, Chs. 22–24] require additional properties of F . Our choice in [4] and in the present paper is

$$F(t) := A(t^{-1}), \quad A \in \mathbb{C} \langle\langle z \rangle\rangle^{n \times n} \supset \mathbb{C}(z)^{n \times n}, \quad t > \sigma(A). \quad (14)$$

A nonzero A admits a unique representation

$$A(z) = z^{-k} B(z), \quad k \in \mathbb{Z}, \quad B(z) = \sum_{i=0}^{\infty} B_i z^i \in \mathbb{C} \langle\langle z \rangle\rangle^{n \times n}, \quad B_i \in \mathbb{C}^{n \times n}, \quad B_0 \neq 0, \quad (15)$$

where the exponent $-k$ is chosen for notational convenience in Theorem 1.1.

Theorem 1.1 Consider a matrix $A(z) \in \mathbb{C} \langle\langle z \rangle\rangle^{n \times n}$ and the state space system defined by the data from (9) to (10).

- (i) If $A = \sum_{i=0}^{\infty} A_i z^i$ is a power series then the system is e.s. if and only if all eigenvalues of A_0 have absolute value < 1 .
- (ii) If $k > 0$ and B_0 is not nilpotent in (15) then the system is not e.s. and indeed

$$\exists t_0 > \sigma(A) \forall t_1 \geq t_0 : \sup_{t \geq t_1} \|\Phi_A(t, t_1)\| = \infty, \quad (16)$$

i.e., the system is unstable.

(iii) Assume $k > 0$ and B_0 nilpotent in (15) and

$$\begin{aligned}\det(B(z)) &= b_\ell z^\ell c(z), \quad \ell \geq 1, \quad 0 \neq b_\ell \in \mathbb{C}, \\ c(z) &= 1 + c_1 z + c_2 z^2 + \cdots \in \mathbb{C} \langle z \rangle.\end{aligned}\quad (17)$$

If $kn > \ell$ then (16) holds and the system is not e.s.

The significance of Theorem 1.1 for arbitrary autonomous behaviors instead of state space behaviors follows from Corollary 3.11.

Example 1.2 That B_0 in item (ii) of Theorem 1.1 is not nilpotent cannot be omitted. To see this consider the nilpotent matrix $B_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2 \times 2}$. For $0 \neq \lambda \in \mathbb{C}$ and $\rho := |\lambda| = e^{-\alpha} > 0$, $\alpha \in \mathbb{R}$, define

$$\begin{aligned}A(z) &:= z^{-1} B(z), \quad B(z) := B_0 + z\lambda \text{id}_2 \implies A(t^{-1}) = B_0 t + \lambda \text{id}_2 = \begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix} \\ \implies \Phi_A(t, t_0) &= \begin{pmatrix} \lambda^{t-t_0} & \lambda^{t-t_0-1} \sum_{i=t_0}^{t-1} i \\ 0 & \lambda^{t-t_0} \end{pmatrix}, \quad \det(B(z)) = \lambda^2 z^2, \quad \det(A(t^{-1})) = \lambda^2.\end{aligned}\quad (18)$$

For $\rho \geq 1$ the sequence $\Phi_A(t, t_0)$ does not converge to zero and therefore the system is not e.s. The sum $\sum_{i=t_0}^{t-1} i$ grows polynomially. If $\rho = e^{-\alpha} < 1$ or $\alpha > 0$ the transition matrix $\Phi_A(t, t_0)$ decreases exponentially with a decay factor $e^{-\alpha'(t-t_0)}$ for every α' with $0 < \alpha' < \alpha$. So the system defined by A is e.s. for $|\lambda| = e^{-\alpha} < 1$.

Remark 1.3 The ring $\mathbb{C} \langle z \rangle$ is defined by *analytic* conditions on the coefficients of the power series that imply its good *algebraic* properties. These are inherited by \mathbf{K} . Also the e.s. of an autonomous behavior is defined by *analytic* conditions on its trajectories [4, Def. 1.7]. In contrast, the construction of the category of behaviors and the derivation of the module-behavior duality proceed *algebraically*. This explains the necessity for both *analytic* and *algebraic* arguments in [4] and the present paper.

Notations and abbreviations $\mathbf{D}(\rho) := \{z \in \mathbb{C}; |z| < \rho\}$, $\rho > 0$, e.s. = exponentially stable, exponential stability, f.g. = finitely generated, LTV = linear time-varying, resp. = respectively, u.e.s. = uniformly e.s., w.e.s. = weakly e.s., $X^{p \times q}$ = set of $p \times q$ -matrices with entries in X , $X^{1 \times q}$ = rows, $X^q := X^{q \times 1}$ = columns, $X^{\bullet \times \bullet} := \bigcup_{p, q \geq 0} X^{p \times q}$.

2 The proof of Theorem 1.1

(i) Since A is a power series we can write

$$\begin{aligned}A(z) &= A_0 + zC(z) \in \mathbb{C} \langle z \rangle^{n \times n}, \\ C(z) &= A_1 + A_2 z + \cdots \in \mathbb{C} \langle z \rangle^{n \times n} \\ \implies A(t^{-1}) &= A_0 + t^{-1} C(t^{-1}), \quad t \geq t_0 > \sigma(A).\end{aligned}\quad (19)$$

The function $C(t^{-1})$ is bounded for $t \geq t_0$ and therefore $t^{-1} C(t^{-1})$ is a disturbance term that is small for large t .

- (a) If $|\lambda| < 1$ for all eigenvalues of A_0 the system $x(t+1) = A_0x(t)$, $t \geq t_0$, is uniformly exponentially stable (u.e.s.). According to [15, Thm. 24.7], [4, Cor. 3.17] the equation $x(t+1) = A(t^{-1})x(t)$, $t \geq t_0$, is also u.e.s. and therefore e.s.
- (b) Assume that A_0 has an eigenvalue λ with $|\lambda| > 1$. According to [4, Thm. 3.21] the system is exponentially unstable and, in particular,

$$\begin{aligned} \exists t_0 > \sigma(A) \exists \rho > 1 \forall t \geq t_1 \geq t_0 : \|\Phi_A(t, t_1)\| \\ \geq \rho^{t-t_1} \implies \sup_{t \geq t_1} \|\Phi_A(t, t_1)\| = \infty. \end{aligned} \quad (20)$$

This implies that the system $x(t+1) = A(t^{-1})x(t)$ is not e.s.

- (c) Assume that A_0 has an eigenvalue λ with $|\lambda| = 1$ and that the system $x(t+1) = A(t^{-1})x(t)$, $t > \sigma(A)$, is e.s. By (13)

$$\begin{aligned} \exists t_0 > \sigma(A) \exists \alpha > 0, \rho := e^{-\alpha} < 1, \exists \text{ p.g. } \varphi \in \mathbb{C}^{t_0+\mathbb{N}}, \varphi > 0, \\ \forall t \geq t_1 \geq t_0 : \|\Phi_A(t, t_1)\| \leq \varphi(t_1)\rho^{t-t_1}. \end{aligned} \quad (21)$$

Now consider the modified system

$$\begin{aligned} y(t+1) &= e^\alpha A(t^{-1})y(t) = \rho^{-1}A(t^{-1})y(t), \quad t > \sigma(A), \quad \text{with} \\ \rho^{-1}A(z) &= (\rho^{-1}A_0) + z(\rho^{-1}C(z)). \end{aligned} \quad (22)$$

The matrix $\rho^{-1}A_0$ has the eigenvalue $\rho^{-1}\lambda$ with $|\rho^{-1}\lambda| = \rho^{-1} = e^\alpha > 1$. From (b) we infer

$$\begin{aligned} \exists t_2 \geq t_1 \forall t \geq t_3 \geq t_2 : \sup_{t \geq t_3} \|\Phi_{\rho^{-1}A}(t, t_3)\| = \infty. \quad \text{But} \\ \Phi_{\rho^{-1}A}(t, t_3) &= \rho^{-(t-t_3)}\Phi_A(t, t_3) \\ \implies \|\Phi_{\rho^{-1}A}(t, t_3)\| &= \rho^{-(t-t_3)}\|\Phi_A(t, t_3)\| \\ &\leq \rho^{-(t-t_3)}(\varphi(t_3)\rho^{t-t_3}) = \varphi(t_3). \end{aligned} \quad (23)$$

The first and the last line of (23) are in contradiction and therefore $x(t+1) = A(t^{-1})x(t)$ cannot be e.s.

This completes the proof of part (i) of the theorem.

- (ii) Recall that a square complex matrix is nilpotent if and only if 0 is its only eigenvalue. Under the conditions of (ii) the matrix B_0 has a nonzero eigenvalue λ . Choose $\rho > |\lambda|^{-1}$ so that $|\rho\lambda| > 1$ for the eigenvalue $\rho\lambda$ of the matrix ρB_0 .

According to (i)(b) ρB is exponentially unstable and indeed

$$\begin{aligned} \exists t_0 > \sigma(A) = \sigma(\rho B) \forall t \geq t_1 \geq t_0 : \sup_{t \geq t_1} \|\Phi_{\rho B}(t, t_1)\| = \infty. \text{ But} \\ A(t^{-1}) = t^k \rho^{-1}(\rho B)(t^{-1}), \quad \Gamma(t, t_1) := (t - 1) * \dots * t_1 \\ \implies \forall t \geq t_1 \geq t_0 : \Phi_A(t, t_1) = \Gamma(t, t_1)^k \rho^{-(t-t_1)} \Phi_{\rho B}(t, t_1) \\ \implies \|\Phi_A(t, t_1)\| = \Gamma(t, t_1)^k \rho^{-(t-t_1)} \|\Phi_{\rho B}(t, t_1)\|. \text{ Further} \\ \Gamma(t, t_1)^k \rho^{-(t-t_1)} = \frac{(t-1)^k}{\rho} * \dots * \frac{t_1^k}{\rho} \xrightarrow{t \rightarrow \infty} \infty \\ \implies \forall t_1 \geq t_0 : \sup_{t \geq t_1} \|\Phi_A(t, t_1)\| = \infty. \end{aligned} \quad (24)$$

(iii) Under the condition of Theorem 1.1, (iii), choose $t_0 > \sigma(A)$ such that $|c(t^{-1})| \geq 1/2$ for $t \geq t_0$. The determinant of

$$\begin{aligned} A(z) = z^{-k} B(z) \text{ is } \det(A(z)) = z^{-kn} \det(B(z)) = b_\ell z^{-(kn-\ell)} c(z) \\ \implies \forall t \geq t_0 : |\det(A(t^{-1}))| = |b_\ell| t^{kn-\ell} |c(t^{-1})| \geq (|b_\ell|/2) t^{kn-\ell} \\ \implies \forall t \geq t_1 \geq t_0 : |\det(\Phi_A(t, t_1))| \geq \Gamma(t, t_1)^{kn-\ell} (|b_\ell|/2)^{t-t_1} \xrightarrow{t \rightarrow \infty} \infty \end{aligned} \quad (25)$$

where the last implication follows as in (24) due to $kn - \ell > 0$. If the sequence $\|\Phi_A(t, t_1)\|$, $t \geq t_1$, was bounded so would be the sequence of determinants $|\det(\Phi_A(t, t_1))|$.

3 Laurent coefficients

We explain the basic notions of a variant of the theory from [4] since we use the difference field $\mathbf{K} = \mathbb{C} \langle\langle z \rangle\rangle$ instead of the field $\mathbb{C}(z) \subset \mathbf{K}$ of rational functions in [4]. Recall $W(t_0) = \mathbb{C}^{t_0+\mathbb{N}}$ for $t_0 \in \mathbb{N}$ from (7). The space $\mathbb{C}^{t_0+\mathbb{N}} = W(t_0)$ is also a difference \mathbb{C} -algebra with the componentwise multiplication and the shift algebra homomorphism

$$\Phi_d : \mathbb{C}^{t_0+\mathbb{N}} \rightarrow \mathbb{C}^{t_0+\mathbb{N}}, \quad c \mapsto \Phi_d(c), \quad \Phi_d(c)(t) = c(t+1), \quad t \geq t_0. \quad (26)$$

It gives rise to the *noncommutative skew-polynomial algebra of difference operators* [9, Section 1.2.3], [4, (20)]

$$\mathbf{B}(t_0) := \mathbb{C}^{t_0+\mathbb{N}}[s; \Phi_d] := \bigoplus_{j=0}^{\infty} \mathbb{C}^{t_0+\mathbb{N}} s^j, \quad sc = \Phi_d(c)s, \quad c \in \mathbb{C}^{t_0+\mathbb{N}}. \quad (27)$$

The space $W(t_0)$ is a left $\mathbf{B}(t_0)$ -module with the action $f \circ w$ for $f = \sum_{j=0}^{\infty} f_j s^j \in \mathbf{B}(t_0)$ and $w \in W(t_0)$ [4, (21)], defined by

$$(f \circ w)(t) := \sum_j f_j(t)w(t+j), \quad t \geq t_0. \quad (28)$$

Of course, almost all, i.e., up to finitely many, f_j are zero so that the sums \sum_j are actually finite, here and in later occurrences. As usual the action is extended to the action $R \circ w$ [4, (22)] of a matrix

$$R = \sum_j R_j s^j \in \mathbf{B}(t_0)^{p \times q}, \quad R_j \in \left(\mathbb{C}^{t_0+\mathbb{N}}\right)^{p \times q}, \quad \text{on } w \in W(t_0)^q \text{ by} \quad (29)$$

$$(R \circ w)(t) := \sum_j R_j(t)w(t+j), \quad t \geq t_0.$$

The *behavior or solution space* defined by R is

$$\mathcal{B}(R, t_0) := \{w \in W(t_0)^q; R \circ w = 0\}. \quad (30)$$

For $\sigma > 0$ the algebra $C^\infty(\sigma, \infty)$ is also a difference algebra with the algebra endomorphism

$$\Phi_s : C^\infty(\sigma, \infty) \rightarrow C^\infty(\sigma, \infty), \quad \Phi_s(f)(t) := f(t+1). \quad (31)$$

It gives rise to the skew-polynomial algebra

$$\mathbf{A}_s(\sigma) := C^\infty(\sigma, \infty)[s; \Phi_s] := \bigoplus_{j \in \mathbb{N}} C^\infty(\sigma, \infty)s^j, \quad sf = \Phi_s(f)s, \quad f \in C^\infty(\sigma, \infty). \quad (32)$$

For $t_0 > \sigma$ the map

$$\Psi_s : C^\infty(\sigma, \infty) \rightarrow \mathbb{C}^{t_0+\mathbb{N}}, \quad f \mapsto (f(t))_{t \geq t_0}, \quad (33)$$

is a difference algebra homomorphism since $\Phi_d \Psi_s = \Psi_s \Phi_s$ and therefore its extension

$$\Psi_s : \mathbf{A}_s(\sigma) = C^\infty(\sigma, \infty)[s; \Phi_s] \rightarrow \mathbf{B}(t_0) = \mathbb{C}^{t_0+\mathbb{N}}[s; \Phi_d],$$

$$\Psi_s \left(\sum_j f_j s^j \right) := \sum_j \Psi_s(f_j) s^j, \quad (34)$$

(denoted with the same letter) is an algebra homomorphism. The algebras $C^\infty(\sigma, \infty)$ and $\mathbb{C}^{t_0+\mathbb{N}}$ are not noetherian and have many zero-divisors and therefore very little is known about the rings of difference operators from (27) to (32) and their modules.

Therefore we restrict the time-varying coefficients of discrete difference equations to sequences

$$a(t^{-1}) = t^{-k} \sum_{i=0}^{\infty} b_i t^{-i}, \quad a = z^k b, \quad b = \sum_{i=0}^{\infty} b_i z^i \in \mathbb{C} \langle z \rangle, \quad (35)$$

where t is chosen sufficiently large as explained in Lemma 3.1, for instance $t > \sigma(a) = \rho(a)^{-1}$. In the latter case we have $(t+1)^{-1} = \frac{t^{-1}}{1+t^{-1}} < \rho(a)$ and

$$\begin{aligned} \forall t > \sigma(a) : a((t+1)^{-1}) &= a\left(t^{-1}(t^{-1}+1)^{-1}\right) \\ &= (t+1)^{-k} \sum_{i=0}^{\infty} b_i (1+t)^{-i} = \left(\frac{t^{-1}}{t^{-1}+1}\right)^k \sum_{i=0}^{\infty} b_i \left(\frac{t^{-1}}{t^{-1}+1}\right)^i. \end{aligned} \quad (36)$$

This suggests to make $\mathbb{C} \langle z \rangle$ a difference field [16, Ex. 1.2], [4, §4.7] via the field automorphism

$$\Phi : \mathbb{C} \langle z \rangle \xrightarrow{\cong} \mathbb{C} \langle z \rangle, \quad \Phi(z) = z(1+z)^{-1}, \quad \Phi(z^{-1}) = z^{-1} + 1. \quad (37)$$

If

$$\begin{aligned} a &= z^k b, \quad b = \sum_{i=0}^{\infty} b_i z^i \in \mathbb{C} \langle z \rangle \quad \text{then } (1+z)^{-i} \\ &= \sum_{j=0}^{\infty} \binom{-i}{j} z^j \in \mathbb{C} \langle z \rangle \\ \text{and } \Phi(a) &:= a\left(\frac{z}{1+z}\right) = \left(\frac{z}{1+z}\right)^k \sum_{i=0}^{\infty} b_i \left(\frac{z}{1+z}\right)^i \\ &= z^k (1+z)^{-k} \sum_{i=0}^{\infty} b_i z^i (1+z)^{-i}. \end{aligned} \quad (38)$$

The corresponding skew-polynomial algebra of difference operators is

$$\mathbf{A} := \mathbf{K}[s; \Phi] = \oplus_{j \in \mathbb{N}} \mathbf{K} s^j, \quad sa = \Phi(a)s. \quad (39)$$

The \mathbb{C} -algebra \mathbf{A} is a noncommutative euclidean domain [9, §1.2], especially principal, and the f.g. left \mathbf{A} -modules are precisely known [9, Thm. 1.2.9, § 5.7., Cor. 5.7.19]. The operators in \mathbf{A} have the form $f := \sum_j f_j s^j \in \mathbf{A}$, $f_j \in \mathbf{K}$, where almost all f_j are zero. We define $\rho(f) := \min_j \{\rho(f_j); f_j \neq 0\}$, $\sigma(f) := \rho(f)^{-1}$.

In the sequel we make use of the equation $\Phi(a)(t^{-1}) = a((t+1)^{-1})$ for $t > \sigma(a)$. Since $\rho(\Phi(a))$ may be smaller than $\rho(a)$, cf. Example 3.3, the left side of this equation

is not defined a priori. To solve this problem we introduce difference subalgebras $\mathbf{K}(\rho) \subset \mathbf{K}$, $\rho > 0$, such that the map $\Psi_\rho : \mathbf{K}(\rho) \rightarrow C^\infty(\rho^{-1}, \infty)$, $a \mapsto a(t^{-1})$, is a well-defined difference algebra homomorphism. We need the following detour: For an open set $U \subseteq \mathbb{C}$ let $\mathcal{O}(U)$ denote the \mathbb{C} -algebra of holomorphic functions in U . So any $a \in \mathbb{C} \ll z \gg$ defines the holomorphic function $a(z) \in \mathcal{O}(\mathbf{D}(\rho(a)) \setminus \{0\})$. In general, this can be extended to larger connected open sets. For $\rho > 0$ we define the subset $\mathbf{K}(\rho) \subset \mathbf{K}$ as follows: An element $a \in \mathbf{K}$ belongs to $\mathbf{K}(\rho)$ if there is an open connected neighborhood $U(a)$ of 0 and a holomorphic function $f \in \mathcal{O}(U(a) \setminus \{0\})$ such that $[0, \rho) \subset U(a)$ and $f(z) = a(z)$ for $z \neq 0$ near 0. In other terms, the germ of f at 0 is a . Since $U(a) \setminus \{0\}$ is connected the function f is unique with these properties, due to the *identity theorem*. The $\mathbf{K}(\rho)$ obviously satisfy

$$a \in \mathbf{K}(\rho(a)), \text{ hence } \mathbf{K} = \bigcup_{\rho > 0} \mathbf{K}(\rho) \text{ and } \forall \rho_1 \leq \rho_2 : \mathbf{K}(\rho_2) \subseteq \mathbf{K}(\rho_1). \quad (40)$$

All entire functions $a \in \mathcal{O}(\mathbb{C}) \subset \mathbb{C} \ll z \gg$ and z^{-1} belong to all $\mathbf{K}(\rho)$.

Definition and Lemma 3.1 1. The value $a(t^{-1}) := f(t^{-1})$ for $t > \rho^{-1}$ is independent of the choice of the extension $f \in \mathcal{O}(U(a) \setminus \{0\})$ of a .
2. The set $\mathbf{K}(\rho)$, $\rho > 0$, is a subalgebra of \mathbf{K} , i.e., additively and multiplicatively closed, and the map

$$\Psi_\rho : \mathbf{K}(\rho) \rightarrow C^\infty(\rho^{-1}, \infty), \quad a \mapsto a(t^{-1}), \quad (41)$$

is an algebra monomorphism

Proof 1. Let $f_i \in \mathcal{O}(U_i \setminus \{0\})$, $i = 1, 2$, be two such extensions. The (open) connected component U_3 of $U_1 \cap U_2$ containing 0 also contains $[0, \rho)$. Since f_1 and f_2 are holomorphic on $U_3 \setminus \{0\}$ and extend a and since $U_3 \setminus \{0\}$ is connected we conclude $f_1|_{U_3 \setminus \{0\}} = f_2|_{U_3 \setminus \{0\}}$ and hence $f_1(t^{-1}) = f_2(t^{-1})$ for $t > \rho^{-1}$.
2. (a) For $a_1, a_2 \in \mathbf{K}(\rho)$ the intersection $U(a_1) \cap U(a_2)$ is an open neighborhood of 0 and contains $[0, \rho)$. Let f_i denote the holomorphic extensions of the a_i to $U(a_i) \setminus \{0\}$ and U_3 the (open) connected component of $U(a_1) \cap U(a_2)$ containing 0 and hence $[0, \rho)$. The function $f_1 + / * f_2$ is holomorphic on $(U(a_1) \cap U(a_2)) \setminus \{0\}$ and hence on $U_3 \setminus \{0\}$ and obviously coincides with $a_1 + / * a_2$ near zero, hence $a_1 + / * a_2 \in \mathbf{K}(\rho)$.

For $t > \rho^{-1}$ this implies

$$\begin{aligned} (a_1 + / * a_2)(t^{-1}) &:= (f_1 + / * f_2)(t^{-1}) = f_1(t^{-1}) + / * f_2(t^{-1}) \\ &= a_1(t^{-1}) + / * a_2(t^{-1}). \end{aligned}$$

Hence $\mathbf{K}(\rho)$ is a subalgebra of \mathbf{K} and Ψ_ρ is an algebra homomorphism.

(2)(b) Ψ_ρ is injective: If $a(t^{-1}) = f(t^{-1}) = 0$ for $t > \rho^{-1}$ the holomorphic function f is zero on $(0, \rho)$. By means of the identity theorem this implies $f = 0$ and $a = 0$. \square

Lemma 3.2 For all $a \in \mathbf{K}$ and $m \in \mathbb{N}$ the following assertions hold:

1. For all $a \in \mathbf{K}$: $\Phi^m(a) = a(z(1 + mz)^{-1})$.
2. For $\rho > 0$, $m \in \mathbb{N}$ and $a \in \mathbf{K}(\rho)$:

$$\begin{aligned} \Phi^m(a) \in \mathbf{K}(\rho_1) \subset \mathbf{K}(\rho), \quad \rho_1 := \begin{cases} \infty & \text{if } m\rho \geq 1 \\ \rho(1 - m\rho)^{-1} & \text{if } m\rho < 1 \end{cases} \geq \rho, \text{ and} \\ \forall t > \rho_1^{-1} = \begin{cases} 0 & \text{if } \rho m \geq 1 \\ \rho^{-1} - m & \text{if } \rho m < 1 \end{cases} : \Phi^m(a)(t^{-1}) = a((t + m)^{-1}). \end{aligned} \quad (42)$$

With Lemma 3.1 this implies that $\mathbf{K}(\rho)$ is a difference subalgebra of \mathbf{K} , i.e. $\Phi(\mathbf{K}(\rho)) \subseteq \mathbf{K}(\rho)$. Moreover Ψ_ρ is a morphism of difference algebras, i.e.,

$$\Phi_s \Psi_\rho = \Psi_\rho \Phi \text{ or } \forall t > \rho^{-1} : \Phi(a)(t^{-1}) = a((t + 1)^{-1}).$$

Proof 1. The equation follows by induction from

$$\begin{aligned} \Phi^{m+1}(a) &= \Phi^m(a)(z(1 + z)^{-1}) \\ &= a\left(\frac{z}{1 + z} \frac{1}{1 + mz(1 + z)^{-1}}\right) = a\left(\frac{z}{1 + (m + 1)z}\right). \end{aligned}$$

2. Let $f \in \mathcal{O}(U(a) \setminus \{0\})$ be a holomorphic extension of a with $U(a) \supset [0, \rho)$. Consider the projective line $\overline{\mathbb{C}} = \mathbb{C} \uplus \{\infty\}$. For $m \geq 0$ there are the inverse biholomorphic maps

$$\overline{\mathbb{C}} \cong \overline{\mathbb{C}}, \quad z = w(1 - mw)^{-1} \leftrightarrow w = z(1 + mz)^{-1}. \quad (43)$$

They induce the inverse biholomorphic maps

$$\begin{aligned} \overline{\mathbb{C}} \setminus \{-1/m, \infty\} &= \mathbb{C} \setminus \{-1/m\} \cong \overline{\mathbb{C}} \setminus \{1/m, \infty\} = \mathbb{C} \setminus \{1/m\} \text{ and} \\ V &:= \left\{ z \in \mathbb{C}; z(1 + mz)^{-1} \in U(a) \right\} \cong U(a) \setminus \{1/m\} \subset U(a). \end{aligned} \quad (44)$$

Since $U(a)$ and $U(a) \setminus \{1/m\}$ are open and connected the set V is a connected open neighborhood of 0. The function $f(z(1 + mz)^{-1})$ is holomorphic on $V \setminus \{0\}$ and coincides with $\Phi^m(a)(z) = a(z(1 + mz)^{-1})$ for small $z \neq 0$. This means that it is a holomorphic extension of $\Phi^m(a)$ or that $\Phi^m(a)$ is its germ at 0. For $z > 0$ the following equivalences hold:

$$z(1 + mz)^{-1} \in (0, \rho) \iff z(1 - m\rho) < \rho \iff z \in (0, \rho_1).$$

Hence

$$\begin{aligned} \forall z \in (0, \rho_1) : m^{-1} \neq z(1 + mz)^{-1} \in (0, \rho) \subset U(a) \\ \implies z \in V \implies (0, \rho_1) \subset V \implies \Phi^m(a) \in \mathbf{K}(\rho_1) \subseteq \mathbf{K}(\rho). \end{aligned} \quad (44)$$

For $t > \rho_1^{-1} = \begin{cases} 0 & \text{if } \rho m \geq 1 \\ \rho^{-1} - m & \text{if } m\rho < 1 \end{cases}$ or $t^{-1} < \rho_1$ this implies

$$(t+m)^{-1} = \frac{t^{-1}}{1+mt^{-1}} < \rho \text{ and}$$

$$\Phi^m(a)(t^{-1}) = f\left(\frac{t^{-1}}{1+mt^{-1}}\right) = a\left(\frac{t^{-1}}{1+mt^{-1}}\right) = a((t+m)^{-1}).$$

□

Example 3.3 Let

$$m := 1, \quad 0 < \alpha < \rho < 1 \text{ and } f(z) := ((z + \alpha)(z - \rho))^{-1} \in \mathcal{O}(\mathbb{C} \setminus \{-\alpha, \rho\}).$$

For the germ $a \in \mathbb{C} \langle z \rangle$ of f at 0 we get

$$\begin{aligned} \rho(a) &= \alpha, \quad (0, \rho) \subset U(a) := \mathbb{C} \setminus \{-\alpha, \rho\}, \quad \forall t > \rho^{-1} : a(t^{-1}) = f(t^{-1}), \\ f\left(\frac{z}{1+z}\right) &= -\alpha^{-1} \rho^{-1} (1+z)^2 \left(1 + \frac{1+\alpha}{\alpha} z\right)^{-1} (1 - \rho_1^{-1} z)^{-1} \\ &\in \mathcal{O}\left(\mathbb{C} \setminus \left\{-\frac{\alpha}{1+\alpha}, \rho_1\right\}\right) \\ \rho(\Phi(a)) &= \frac{\alpha}{1+\alpha} < \alpha < \rho < \rho_1 := \frac{\rho}{1-\rho}, \quad (0, \rho_1) \subset \mathbb{C} \setminus \left\{-\frac{\alpha}{1+\alpha}, \rho_1\right\} \\ \forall t > \rho_1^{-1} &= \rho^{-1} - 1 : t+1 > \rho^{-1} \text{ and} \\ \Phi(a)(t^{-1}) &= f\left(\frac{t^{-1}}{1+t^{-1}}\right) = f((t+1)^{-1}) = a((t+1)^{-1}). \end{aligned}$$

Remark 3.4 The preceding example can be generalized. Assume that $a \in \mathbb{C} \langle z \rangle$ can be extended to $f \in \mathcal{O}(\mathbb{C} \setminus S)$ where S is a discrete closed subset of \mathbb{C} and S is the set of singularities of f . So a is the germ of f at zero, $U := \mathbb{C} \setminus S$ is connected and both S and f are uniquely determined by a . Therefore $a(z) := f(z)$, $z \in U$, is well-defined. In particular, $a(t^{-1})$ is well-defined for $t \in \mathbb{N}$ and $t^{-1} \in (0, \rho) \cap U$. The inclusion $(0, \rho) \subset U$ is not required in this case, but was essential for the definition and properties of $\mathbf{K}(\rho)$.

The Lemmas 3.1 and 3.2 imply that

$$\mathbf{A}(\rho) := \mathbf{K}(\rho)[s; \Phi] = \bigoplus_{i \in \mathbb{N}} \mathbf{K}(\rho)s^i \subset \mathbf{A} = \mathbf{K}[s; \Phi] \quad (45)$$

is a subalgebra of \mathbf{A} and that the extended map, denoted by the same letter,

$$\begin{aligned} \Psi_\rho : \mathbf{A}(\rho) &= \mathbf{K}(\rho)[s; \Phi] \rightarrow \mathbf{A}_s(\rho^{-1}) \\ &= \mathcal{C}^\infty(\rho^{-1}, \infty)[s; \Phi_s], \quad a(z)s^j \mapsto a(t^{-1})s^j, \end{aligned} \quad (46)$$

is an algebra monomorphism. Since any $f = \sum_j f_j s^j$ belongs to $\mathbf{K}(\rho(f))$, $\rho(f) = \min \{\rho(f_j); f_j \neq 0\}$, we again have $\mathbf{A} = \bigcup_{\rho>0} \mathbf{A}(\rho)$.

For $t_0 > \rho^{-1}$ we compose Ψ_ρ and Ψ_s and obtain the algebra homomorphism

$$\begin{aligned} \Psi &:= \Psi_s \Psi_\rho : \mathbf{K}(\rho) \xrightarrow{\Psi_\rho} \mathbb{C}^\infty(\rho^{-1}, \infty) \xrightarrow{\Psi_s} \mathbb{C}^{t_0+\mathbb{N}} \\ &\quad \bigcap \quad \bigcap \quad \bigcap \\ \Psi &:= \Psi_s \Psi_\rho : \mathbf{A}(\rho) \xrightarrow{\Psi_\rho} \mathbf{A}_s(\rho^{-1}) \xrightarrow{\Psi_s} \mathbf{B}(t_0) \\ &\quad a(z)s^j \mapsto a(t^{-1})s^j \mapsto (a(t^{-1}))_{t \geq t_0} s^j \end{aligned} \quad (47)$$

Lemma 3.5 Ψ is injective.

Proof Since Ψ is a homomorphism and z is a unit in $\mathbf{K}(\rho)$ it suffices to show that for a power series $a \in \mathbf{K}(\rho)$ the equation $\Psi(a) = (a(t^{-1}))_{t \geq t_0} = 0$ implies $a = 0$. This holds by the *identity theorem* since 0 is an accumulation point of the sequence t^{-1} . \square

Definition and Corollary 3.6 For $\rho > 0$ and $t_0 > \rho^{-1}$ the algebra monomorphism Ψ and the left $\mathbf{B}(t_0)$ -module structure of $W(t_0) = \mathbb{C}^{t_0+\mathbb{N}}$ from (28) imply the action \circ of $\mathbf{A}(\rho)$ on $W(t_0)$ defined by

$$\begin{aligned} f \circ w &:= \Psi(f) \circ w, \quad f = \sum_j f_j s^j \in \mathbf{A}(\rho), \quad w \in W(t_0), \\ \forall t \geq t_0 > \rho^{-1} : (f \circ w)(t) &= \sum_j f_j(t^{-1})w(t+j) \end{aligned} \quad (48)$$

This action makes $W(t_0)$ a left $\mathbf{A}(\rho)$ -module, in particular $(fg) \circ w = f \circ (g \circ w)$ holds for $f, g \in \mathbf{A}(\rho)$ and $w \in \mathbb{C}^{t_0+\mathbb{N}}$, $t_0 > \rho^{-1}$.

The homomorphisms Φ, Φ_d, Ψ are extended to matrices componentwise. For a matrix $R = \sum_j R_j s^j \in \mathbf{A}^{p \times q}$, $R_j \in \mathbf{K}^{p \times q}$, we define, cf. (6),

$$\rho(R) := \min \{\rho(R_j); R_j \neq 0\}, \quad \sigma(R) = \rho(R)^{-1}, \quad \text{and obtain } R \in \mathbf{A}(\rho(R))^{p \times q}. \quad (49)$$

A matrix $R \in \mathbf{A}(\rho)^{p \times q}$ and $t_0 > \rho^{-1}$ give rise to

$$\Psi(R) = \sum_j \left(R_j(t^{-1}) \right)_{t \geq t_0} s^j \in \mathbf{B}(t_0)^{p \times q} = \bigoplus_j \left(\mathbb{C}^{t_0+\mathbb{N}} \right)^{p \times q} s^j$$

and the solution space or behavior

$$\begin{aligned} \mathcal{B}(R, t_0) &:= \{w \in W(t_0)^q; R \circ w = 0\} = \{w \in W(t_0)^q; \Psi(R) \circ w = 0\} \\ &= \left\{ w \in W(t_0)^q; \forall t \geq t_0 : \sum_j R_j(t^{-1})w(t+j) = 0 \right\}. \end{aligned} \quad (50)$$

As in Definition and Corollary 3.6 the equation

$$(RS) \circ w = R \circ (S \circ w), \quad R, S \in \mathbf{A}(\rho)^{p \times q}, \quad w \in W(t_0)^q, \quad t_0 > \rho^{-1}, \quad (51)$$

holds and is decisive for the duality theory, recalled below.

For $A \in \mathbf{K}(\rho)^{q \times q}$ the state space behavior from (9) now obtains the form

$$\mathcal{B}(s \operatorname{id}_q - A, t_0) = \mathcal{K}(A, t_0) = \left\{ x \in W(t_0)^q; \quad \forall t \geq t_0 : x(t+1) = A(t^{-1})x(t) \right\}. \quad (52)$$

Remark 3.7 Notice that $W(t_0)$ is not an \mathbf{A} -, but only an $\mathbf{A}(\rho)$ -left module for $t_0 > \rho^{-1}$. In contrast to the well-known algebraic structure of \mathbf{A} that of $\mathbf{A}(\rho)$ and its f.g. left modules is unknown. To enable a module-behavior duality between f.g. \mathbf{A} -left modules and behaviors those from (50) have to be modified as in [4, (7), (9)], see (54)–(65) below.

The subspace $\mathbb{C}^{(\mathbb{N})} \subset \mathbb{C}^{\mathbb{N}}$ consists of the signals $w \in \mathbb{C}^{\mathbb{N}}$ with finite support $\operatorname{supp}(w) := \{t \in \mathbb{N}; w(t) \neq 0\}$. The factor space $W(\infty) := \mathbb{C}^{\mathbb{N}}/\mathbb{C}^{(\mathbb{N})}$ is an \mathbf{A} -left module with the action $f \circ (w + \mathbb{C}^{(\mathbb{N})}) := v + \mathbb{C}^{(\mathbb{N})}$ where

$$f = \sum_j f_j s^j \in \mathbf{A}, \quad v(t) := \begin{cases} \sum_j f_j(t^{-1})w(t+j) & \text{if } t > \sigma(f) \\ 0 & \text{if } t \leq \sigma(f) \end{cases}. \quad (53)$$

This signal space $W(\infty)$ was already defined in [16, p. 5]. In [3, Thm. 839] it was shown for the coefficient field $\mathbb{C}(z)$ instead of $\mathbb{C} \ll z \gg$ here that it is a *large injective \mathbf{A} -cogenerator* and thus enables a module-behavior duality. This signal module $W(\infty)$ is unsuitable for the stability theory of LTV systems since the signals $w + \mathbb{C}^{(\mathbb{N})}$ do not have well-defined values $w(t) \in \mathbb{C}$ and in particular no initial value $w(t_0)$. So (uniform) exponential stability of state space behaviors as in (13) or of general behaviors [4, Def. 1.7] cannot be defined.

However, $W(\infty)$ is even a commutative ring since $\mathbb{C}^{(\mathbb{N})}$ is an ideal of $\mathbb{C}^{\mathbb{N}}$ with the componentwise multiplication. The shift on $\mathbb{C}^{\mathbb{N}}$ induces that on $W(\infty)$ and makes it a difference ring. The ring homomorphism

$$\tilde{\Psi} : \mathbf{K} \rightarrow W(\infty), \quad a \mapsto \tilde{\Psi}(a) := \tilde{a} + \mathbb{C}^{(\mathbb{N})}, \quad \tilde{a}(t) := \begin{cases} a(t^{-1}) & \text{if } t > \sigma(a) \\ 0 & \text{if } t \leq \sigma(a). \end{cases}$$

is well-defined, injective and indeed a monomorphism of difference rings. The algebra $W(\infty)$ and the preceding homomorphism can be used instead of $\Psi : \mathbf{K}(\rho) \rightarrow \mathbb{C}^{t_0+\mathbb{N}}$ from (47) to derive the duality theory, recalled below, with different, but similar arguments.

Two behavior families

$$\mathcal{B}_i := (\mathcal{B}(R_i, t_0))_{t_0 > \rho_i^{-1}}, \quad R_i \in \mathbf{A}(\rho_i)^{p_i \times q} \subset \mathbf{A}^{p_i \times q}, \quad i = 1, 2, \quad (54)$$

are called *equivalent*, cf. [4, (7)], if

$$\exists t_1 > \max(\rho_1^{-1}, \rho_2^{-1}) \forall t_2 \geq t_1 : \mathcal{B}(R_1, t_2) = \mathcal{B}(R_2, t_2). \quad (55)$$

Since t_1 can be chosen large one may always assume that $\rho_i = \rho(R_i)$ for $i = 1, 2$. The equivalence class of \mathcal{B}_1 is denoted by $\text{cl}(\mathcal{B}_1)$. The study of this class means to study the behaviors $\mathcal{B}(R_1, t_2)$ for large t_2 . This is appropriate for stability questions where the trajectories $w(t)$ of a behavior are studied for $t \rightarrow \infty$.

If M is a f.g. \mathbf{A} -module with a given list $\mathbf{w} := (\mathbf{w}_1, \dots, \mathbf{w}_q)^\top$ of generators there is the canonical isomorphism

$$\begin{aligned} \mathbf{A}^{1 \times q} / U &\cong M, \quad \xi + U \mapsto \xi \mathbf{w} = \sum_i \xi_i \mathbf{w}_i \text{ where} \\ \xi &= (\xi_1, \dots, \xi_q) \in \mathbf{A}^{1 \times q}, \quad U := \left\{ \xi \in \mathbf{A}^{1 \times q}; \quad \xi \mathbf{w} = 0 \right\}. \end{aligned} \quad (56)$$

Since \mathbf{A} is noetherian the submodule U is f.g. and thus generated by the rows of some matrix $R \in \mathbf{A}^{p \times q}$, i.e., $U = \mathbf{A}^{1 \times p} R$. Since \mathbf{A} is even a principal ideal domain U is free and one may assume that $\dim_{\mathbf{A}}(U) = \text{rank}(R) = p$. The matrix R gives rise to behaviors

$$\mathcal{B}(R, t_0), \quad t_0 > \sigma(R) = \rho(R)^{-1}, \text{ and their class } \text{cl}((\mathcal{B}(R, t_0))_{t_0 > \sigma(R)}). \quad (57)$$

Equation (51) is used to show that this class depends on U only and not on the special choice of R [4, Lemma 2.5]. The class is denoted by

$$\mathcal{B}(U) := \text{cl}((\mathcal{B}(R, t_0))_{t_0 > \sigma(R)}), \quad U = \mathbf{A}^{1 \times p} R \subseteq \mathbf{A}^{1 \times q}, \quad (58)$$

and called *the behavior defined by U* . These behaviors $\mathcal{B}(U)$ were introduced in [4, (9)] for the coefficient field $\mathbb{C}(z) = \mathbb{C}(z^{-1})$ of rational functions instead of $\mathbf{K} \supset \mathbb{C}(z)$. In particular, a matrix $A \in \mathbf{K}^{n \times n}$ gives rise to the solution spaces $\mathcal{B}(s \text{id}_n - A, t_0) = \mathcal{K}(A, t_0)$ from (9) to (52) and the *Kalman state space behavior*

$$\mathcal{B}(U) = \text{cl}((\mathcal{K}(A, t_0))_{t_0 > \sigma(A)}), \quad A \in \mathbf{K}^{n \times n}, \quad U = \mathbf{A}^{1 \times n}(s \text{id}_n - A). \quad (59)$$

Remark 3.8 The following three properties of the coefficient sequences $(a(t^{-1}))_{t > \sigma(a)}$, $a \in \mathbf{K}$, are decisive:

1. Any nonzero $a \in \mathbb{C} < z >$ can be written as $a = a_0 + zc(z)$ where $c(z)$ is bounded for $|z| \leq \rho < \rho(a)$ and hence $a(t^{-1}) = a_0 + t^{-1}c(t^{-1})$ with bounded $c(t^{-1})$ for $t \geq t_0 > \sigma(a)$. For large t the term $t^{-1}c(t^{-1})$ is a small disturbance of the constant a_0 and thus the perturbation results [15, Thm. 24.7], [4, Lemma 3.15] are applicable.
2. The sequences are of at most polynomial growth, indeed

$$\begin{aligned} a &= z^{-m}b, \quad m \geq 0, \quad b \in \mathbb{C} < z > \\ \text{implies } \exists c > 0 \forall t \geq t_0 > \sigma(a) : |a(t^{-1})| &\leq ct^m. \end{aligned} \quad (60)$$

3. The sequences have no zeros for large t , i.e., $\exists t_1 > \sigma(a) \forall t \geq t_1 : a(t^{-1}) \neq 0$.

Result 3.9 (Meta-theorem) *With the obvious necessary modifications all essential notions and results from [4] hold for the coefficient field \mathbf{K} and the behaviors $\mathcal{B}(U)$ defined in (54).*

For the preceding result one checks that the proofs of [4] use the properties of Remark 3.8 only. In particular, there is a canonical definition of behavior morphisms: If $M_i = \mathbf{A}^{1 \times q_i} / U_i$, $U_i = \mathbf{A}^{1 \times p_i} R_i$, $R_i \in \mathbf{A}^{p_i \times q_i}$, $i = 1, 2$, are two modules any \mathbf{A} -linear map $\varphi : M_1 \rightarrow M_2$ has the form [4, (47), (48)]

$$\begin{aligned} \varphi = (\circ P)_{\text{ind}} : M_1 \rightarrow M_2, \quad \xi + U_1 \mapsto \xi P + U_2, \quad \text{where} \\ P \in \mathbf{A}^{q_1 \times q_2}, \quad \exists X \in \mathbf{A}^{p_1 \times p_2} \text{ with } R_1 P = X R_2. \end{aligned} \quad (61)$$

If $R_1, R_2, P, X \in \mathbf{A}(\rho)^{\bullet \times \bullet}$, for instance if $\rho < \min(\rho(R_1), \rho(R_2), \rho(P), \rho(X))$, and if $t_0 > \rho^{-1}$ Eq. 51 implies

$$\begin{aligned} \forall w \in W(t_0)^{q_2} : R_1 \circ (P \circ w) = X \circ (R_2 \circ w) \\ \implies P \circ : \mathcal{B}(R_2, t_0) \rightarrow \mathcal{B}(R_1, t_0), \quad w_2 \mapsto P \circ w_2. \end{aligned} \quad (62)$$

The equivalence class $\mathcal{B}(\varphi) := \text{cl}((P \circ : \mathcal{B}(R_2, t_0) \rightarrow \mathcal{B}(R_1, t_0))_{t_0 > \rho^{-1}})$ is defined in analogy to (54), cf. [4, (33), (50)], and defines the behavior morphism

$$\mathcal{B}(\varphi) : \mathcal{B}(U_2) \rightarrow \mathcal{B}(U_1). \quad (63)$$

The map φ is an isomorphism if and only if $\mathcal{B}(\varphi)$ is one, i.e., if $P \circ : \mathcal{B}(R_2, t_1) \rightarrow \mathcal{B}(R_1, t_1)$ is an isomorphism for sufficiently large t_1 .

Corollary 3.10 (cf. [4, Cor. 2.7, Thm. 1.6.]) *The behaviors $\mathcal{B}(U)$ with these morphisms form an abelian category and the assignment $M = \mathbf{A}^{1 \times q} / U \mapsto \mathcal{B}(U)$ is a categorical duality or contravariant equivalence from f.g. \mathbf{A} -modules M with a given finite list of generators to behaviors.*

A f.g. module $M = \mathbf{A}^{1 \times q_1} / U_1$ is a torsion module, cf. [4, §2.5], if and only if $n := \dim_{\mathbf{K}}(M) < \infty$ or if and only if it is isomorphic to a module in state space form, i.e.,

$$\begin{aligned} M = \mathbf{A}^{1 \times q_1} / U_1 \cong M_2 = \mathbf{A}^{1 \times n} / U_2, \quad U_2 = \mathbf{A}^{1 \times n} (s \text{id}_n - A), \quad A \in \mathbf{K}^{n \times n} \\ \mathcal{B}(U_2) = \text{cl}((\mathcal{K}(A, t_0))_{t_0 > \sigma(A)}) \cong \mathcal{B}(U_1). \end{aligned} \quad (64)$$

The behavior

$$\mathcal{B}(U) = \text{cl}((\mathcal{B}(R, t_0))_{t_0 > \sigma(R)}) \quad \text{with } R \in \mathbf{A}^{p \times q}, \quad U = \mathbf{A}^{1 \times p} R, \quad M = \mathbf{A}^{1 \times q} / U \quad (65)$$

is called *autonomous* if and only if there is $t_1 > \sigma(R)$ such that all trajectories $w \in \mathcal{B}(R, t_2)$, $t_2 \geq t_1$, are determined by the initial vector $(w(t_2), \dots, w(t_2 + d))$

of some fixed length d . This is the case if and only if the module $M = \mathbf{A}^{1 \times q} / U$ is a torsion module [4, Thm. 3.18].

Exponential stability (e.s.) of an autonomous behavior $\mathcal{B}(U)$ and of its torsion module $M = \mathbf{A}^{1 \times q} / U$ [4, Def. 1.7] is defined by a more general version of (13) and preserved by behavior isomorphisms [4, § 3.2]. The state space behavior $\mathcal{B}(U_2)$ in (64) is e.s. if and only if the matrix A is e.s. in the sense of (13) [4, Cor. 3.3].

Corollary 3.11 *In the situation of (64) the general autonomous behavior $\mathcal{B}(U_1)$ is e.s. if and only if the matrix A is e.s. in the sense of (13). The matrix A and an explicit isomorphism $M_1 \cong M_2$ or $\mathcal{B}(U_2) \cong \mathcal{B}(U_1)$ can be computed with the OreModules package [6, 14]. Hence Theorem 1.1 describes an algorithmic test of e.s. for all autonomous behaviors except those for which the assumptions of Theorem 1.1, (ii), (iii), are not satisfied by A .*

Corollary 3.12 ([4, Thms. 1.8, 3.11]) *The e.s. behaviors and modules form Serre categories, i.e., are closed under subobjects, factor objects and extensions.*

4 Puiseux series and weak exponential stability

To a large extent Theorem 1.1 can be extended to the difference field \mathbf{P} of locally convergent Puiseux series, cf. [5, §3.1], where

$$\mathbf{P} := \bigcup_{m \geq 1} \mathbb{C} \langle\langle z^{1/m} \rangle\rangle; \quad \Phi(z) = z(1+z)^{-1}, \quad \Phi(z^{1/m}) = z^{1/m}(1+z)^{-1/m}. \quad (66)$$

and its Φ -invariant Bézout and valuation subdomain

$$\mathbf{L} := \bigcup_{m \geq 1} \mathbb{C} \langle z^{1/m} \rangle. \quad (67)$$

The field \mathbf{P} is the algebraic closure of $\mathbf{K} = \mathbb{C} \langle\langle z \rangle\rangle$ [11]. If m_1 divides m_2 then $z^{1/m_1} = (z^{1/m_2})^{m_2/m_1}$ and hence $\mathbb{C} \langle z^{1/m_1} \rangle \subseteq \mathbb{C} \langle z^{1/m_2} \rangle$. The nonzero elements of $\mathbb{C} \langle\langle z^{1/m} \rangle\rangle$ have the unique form

$$a(z^{1/m}) = z^{k/m} b(z^{1/m}), \quad a = z^k b \in \mathbb{C} \langle\langle z \rangle\rangle, \\ k \in \mathbb{Z}, \quad b = \sum_i b_i z^i \in \mathbb{C} \langle z \rangle, \quad b_0 \neq 0. \quad (68)$$

Such an $a(z^{1/m})$ induces the smooth function $a(t^{-1/m}) = t^{-k/m} \sum_i b_i t^{-i/m}$ on the real interval $(\sigma(a)^m, \infty)$. The coefficient rings give rise to the operator domains

$$\mathbf{L}[s; \Phi] \subseteq \mathbf{B} := \mathbf{P}[s; \Phi] = \oplus_{j \in \mathbb{N}} \mathbf{P}s^j. \quad (69)$$

Again \mathbf{B} is a left and right euclidean domain. A matrix

$$\begin{aligned} R &= \sum_j R_j(z^{1/m})s^j \in \mathbf{B}^{p \times q} \text{ with } R_j \\ &= (R_{j\mu\nu}(z))_{\mu,\nu} \in \mathbb{C}^{<< z >>^{p \times q}} \text{ and} \\ R_j(z^{1/m}) &\in \mathbb{C}^{<< z^{1/m} >>^{p \times q}}, \\ \sigma(R) &:= \max \{ \sigma(R_j); R_j \neq 0 \} = \max_{j,\mu,\nu} \sigma(R_{j\mu\nu}(z)) \text{ (cf. (49))}, \end{aligned} \quad (70)$$

acts on $w \in W(t_0)^q = (\mathbb{C}^q)^{t_0 + \mathbb{N}}$ with $t_0 > \sigma(R)^m$ via

$$\begin{aligned} (R \circ w)(t) &:= \sum_j R_j(t^{-1/m})w(t+j), \quad t_0 > \sigma(R)^m, \\ \mathcal{B}(R, t_0) &:= \{w \in W(t_0)^q; R \circ w = 0\}. \end{aligned} \quad (71)$$

In particular, a matrix $A(z^{1/m}) \in \mathbf{P}^{n \times n}$ induces the state space equation and behaviors

$$\begin{aligned} x(t+1) &= A(t^{-1/m})x(t), \quad t \geq t_0 > \sigma(A)^m, \\ \mathcal{K}(A(z^{1/m}), t_0) &:= \mathcal{B}(s \operatorname{id}_q - A(z^{1/m}), t_0) \\ &= \left\{ x \in W(t_0)^n; x(t+1) = A(t^{-1/m})x(t) \right\}. \end{aligned} \quad (72)$$

Result 4.1 Since \mathbf{P} satisfies the conditions of Remark 3.8 the notions and theorems of [4] also hold for \mathbf{P} like for $\mathbb{C}(z)$ in [4] and for $\mathbf{K} = \mathbb{C}^{<< z >>}$ in Sect. 3.

Corollary 4.2 Consider $A = A(z) \in \mathbb{C}^{<< z >>^{n \times n}}$ and the state space system

$$x(t+1) = A(t^{-1/m})x(t), \quad t > \sigma(A)^m.$$

- (i) If $A(z) = A_0 + A_1 z + \dots \in \mathbb{C}^{< z >^{n \times n}}$ is a power series the system is e.s. if and only if all eigenvalues of A_0 have absolute value < 1 .
- (ii) If $A(z) = z^{-k} B(z)$, $B(z) = \sum_{i=0}^{\infty} B_i z^i$, $k > 0$ and B_0 is not nilpotent then

$$\exists t_0 > \sigma(A)^m \forall t_1 \geq t_0 : \sup_{t \geq t_1} \|\Phi_{A(t^{-1/m})}(t, t_1)\| = \infty \quad (73)$$

and the system is not e.s.

- (iii) If for $A(z)$ as in (ii) the determinant of $\det(B(z))$ has the form $\det(B(z)) = b_\ell z^\ell c(z) \neq 0$ where $c(z) = 1 + \sum_{i=1}^{\infty} c_i z^i$ and $kn - \ell > 0$ then (73) holds and the system is not e.s.

One can weaken the definition of exponential stability to *weak exponential stability* (w.e.s.) as in [5, Def. 2.4]. This is also preserved by behavior isomorphisms. For the state space system (72) w.e.s. holds if and only if

$$\begin{aligned} \exists t_0 > \sigma(A)^m \exists \rho = e^{-\alpha} < 1 \ (\alpha > 0) \exists \mu > 0 \text{ p.g. } \varphi \in \mathbb{C}^{t_0 + \mathbb{N}}, \varphi > 0, \\ \forall t_1 \geq t_0 : \|\Phi_{A(t^{-1/m})}(t, t_1)\| &\leq \varphi(t_1) \rho^{t^\mu - t_1^\mu} = \varphi(t_1) e^{-\alpha(t^\mu - t_1^\mu)}. \end{aligned} \quad (74)$$

The difference to e.s. is the exponent $\mu > 0$. If $\mu < 1$ the factor $\exp(-\alpha t^\mu)$ decreases more slowly than $\exp(-\alpha t)$. Corollary 4.2 obtains a slightly weaker form.

Corollary 4.3 *Corollary 4.2 remains true for w.e.s. instead of e.s. with the following exception in part (i): If $A(z) = \sum_{i=0}^{\infty} A_i z^i \in \mathbb{C} \langle z \rangle^{n \times n}$ then the system is e.s. and hence w.e.s. resp. unstable if $|\lambda| < 1$ for all eigenvalues λ of A_0 resp. $|\lambda| > 1$ for at least one. If $|\lambda| \leq 1$ for all eigenvalues of A_0 and $|\lambda| = 1$ for at least one the system may be w.e.s. in contrast to Corollary 4.2, (i), cf. Example 4.4.*

Example 4.4 (Cf. [16, §6.1, §7.1]) Let $\alpha > 0$, $\rho := e^{-\alpha}$, $\mu = k_0/m$, $0 < k_0 < m$. The signal $x(t) := e^{-\alpha t^\mu} = \rho^{t^\mu}$ satisfies

$$\begin{aligned} x(t+1)/x(t) &= \exp(-\alpha((t+1)^\mu - t^\mu)). \text{ But} \\ (t+1)^\mu - t^\mu &= t^\mu \left((1+t^{-1})^\mu - 1 \right) = (z^{-\mu}((1+z)^\mu - 1)) (t^{-1}) \\ z^{-\mu}((1+z)^\mu - 1) &= z^{-\mu} \sum_{k=1}^{\infty} \binom{\mu}{k} z^k = \sum_{k=1}^{\infty} \binom{\mu}{k} (z^{1/m})^{km-k_0} \in \mathbb{C} \langle z^{1/m} \rangle. \end{aligned} \quad (75)$$

We define

$$\begin{aligned} a(z) &:= \exp \left(-\alpha \sum_{k=1}^{\infty} \binom{\mu}{k} z^{km-k_0} \right) = 1 - \alpha \mu z^{m-k_0} + \dots \in \mathbb{C} \langle z \rangle \\ \implies a(z^{1/m}) &= \exp(-\alpha z^{-\mu}((1+z)^\mu - 1)) \\ &= 1 - \alpha \mu z^{1-\mu} + \dots \in \mathbb{C} \langle z^{1/m} \rangle \\ \implies a(t^{-1/m}) &= \exp(-\alpha((t+1)^\mu - t^\mu)), \quad x(t+1) = a(t^{-1/m})x(t), \\ \implies x(t) &= \exp(-\alpha(t^\mu - t_0^\mu)) x(t_0), \quad |x(t)| = \exp(-\alpha(t^\mu - t_0^\mu)) |x(t_0)|. \end{aligned} \quad (76)$$

This equation is w.e.s. with exponent μ . But $a(0) = 1$ and the equation is not e.s by Corollary 4.2.

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